Algebra II
Mid Term Examination
Solutions

1. Let A, B, D be square matrices of size n and let 0 denote the zero matrix. Prove that

$$det \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = det(A)det(D).$$

Ans. Let

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} & b_{11} & \dots & b_{1n} \\ . & \dots & . & . & \dots & . \\ a_{n1} & \dots & a_{nn} & b_{n1} & \dots & b_{nn} \\ 0 & \dots & 0 & d_{11} & \dots & d_{1n} \\ . & \dots & . & . & \dots & . \\ 0 & \dots & 0 & d_{n1} & \dots & d_{nn} \end{pmatrix}.$$

Let us write

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = S = \begin{pmatrix} s_{11} & \dots & s_{1n} & s_{1\ n+1} & \dots & s_{1\ 2n} \\ . & \dots & . & . & \dots & . \\ s_{n1} & \dots & s_{nn} & s_{n\ n+1} & \dots & s_{n\ 2n} \\ s_{n+1\ 1} & \dots & s_{n+1\ n} & s_{n+1\ n+1} & \dots & s_{n+1\ 2n} \\ . & \dots & . & . & \dots & . \\ s_{2n\ 1} & \dots & s_{2n\ n} & s_{2n\ n+1} & \dots & s_{2n\ 2n} \end{pmatrix}.$$

$det(S) = \sum_\sigma (-1)^\sigma s_{1\ \sigma(1)} s_{2\ \sigma(2)} \dots s_{2n\ \sigma(2n)}$, where σ is an element of the symmetric group S_{2n} and $(-1)^\sigma$ is $+1$ is σ is an even permutation and -1 if σ is an odd permutation. Note that $s_{n+1\ 1}, \dots, s_{n+1\ n}, \dots, s_{2n\ 1}, \dots, s_{2n\ n}$ are all zero. Therefore, right hand side of the above equation will have non-zero contribution only when $\{\sigma(1), \dots, \sigma(n)\} \subseteq \{1, \dots, n\}$ and $\{\sigma(n+1), \dots, \sigma(2n)\} \subseteq \{n+1, \dots, 2n\}$. Hence,

$$det(S) = \big(\sum_\alpha (-1)^\alpha s_{1\ \alpha(1)} \dots s_{n\ \alpha(n)}\big) \ \big(\sum_\beta (-1)^\beta s_{n+1\ \beta(n+1)} \dots s_{2n\ \beta(2n)}\big),$$

where $\alpha \in S_n$ is a permutation of $\{1, \dots, n\}$ and $\beta \in S_n$ is a permutation of $\{n+1, \dots, 2n\}$. Note that $\alpha \circ \beta$ (here composition is juxtaposition) is an element of S_{2n} and since their cycles are disjoint we will have $(-1)^\alpha(-1)^\beta = (-1)^{\alpha\circ\beta}$. Therefore, $det(S) = det(A)det(D)$.

2. Let A be an $n \times n$ matrix with integer entries a_{ij}. Prove that A^{-1} has integer entries if and only if $det(A) = \pm 1$.
Ans. Let us first assume entries of A belong to a commutative ring in which ± 1 is invertible and $det(A) = \pm 1$. Since $det(A)$ is invertible, A^{-1} exists and $A^{-1} = \frac{1}{det(A)} Adj(A)$. Since all the entries of A are integers, all the entries of $Adj(A)$ will also be integers. Hence from the formula it is clear that all the entries of A^{-1} will be integers.

Next we assume A^{-1} exists and all the entries of A^{-1} are integers. Since all the entries of A are integers, all the entries of $Adj(A)$ will also be integers. Hence it follows from the formula $A^{-1} = \frac{1}{det(A)} Adj(A)$ that $det(A)$ must be ± 1.

3. Let i denote an element whose square is -1. Prove that the set $\{a + ib | a, b \in \mathbb{Z}/3\mathbb{Z}\}$ is a field under natural addition and multiplication.
Ans. It is routine to check with respect to natural addition and multiplication the above set forms a commutative ring. It remains to check that every non-zero element in the above set has a multiplicative inverse in the set itself. Let $a + ib$ be a non-zero element, i.e, both a and b are not zero. Therefore the choices for the ordered pair (a, b) could be either of $(0,1), (0,2), (1,1), (1,2), (1,0), (2,0), (2,1)$ and in all these cases $a^2 + b^2 = \pm 1$ in $\mathbb{Z}/3\mathbb{Z}$. Therefore the inverse of $(a+ib)^{-1} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$ belongs to the above set.

4. Let V be a finite dimensional vector space over a field F. Prove that any linearly independent subset of V is a subset of a spanning linearly independent subset of V.
Ans. See *Topics in Algebra*, I.N. Herstein (2nd Ed), Lemma 4.2.5 or *Abstract Algebra*, Dummit and Foote (2nd Ed), Chapter 11, Section 1, Corollary 5.

5. Show that the subset $W = \{(x_1, \ldots, x_n) | x_1 + 2x_2 + \cdots + nx_n = 0\}$ of $\mathbb{R}^n$ is a subspace and find a basis for W.
Ans. Note that if $(x_1, \ldots, x_n) \in W$ and $(y_1, \ldots, y_n) \in W$, then $(x_1 + y_1, \ldots, x_n + y_n) \in W$ and $(\alpha x_1, \ldots, \alpha x_n) \in W$, for all $\alpha \in \mathbb{R}$. This shows W is a subspace of $\mathbb{R}^n$. Also, note that W is the solution space of a single linear homogeneous equation in $\mathbb{R}^n$ and hence W is of dimension $n-1$. Therefore if we can exhibit a set of $n-1$ linearly independent vectors in W that will form a basis for W. Clearly the collection of vectors $(-2, 1, 0, \ldots, 0), (-3, 0, 1, \ldots, 0), \ldots, (-n, 0, 0, \ldots, 1)$ are $n-1$ linearly independent vectors in W which will form a basis.

6. Prove that a square matrix with entries from a field F is invertible if and only if its columns are linearly independent.

Ans. Let us consider a $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{n1} \\ . & \dots & . \\ . & \dots & . \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = (A_1, \dots, A_n)$$

where $A_1, \dots, A_n$ are n columns of A. An $n \times n$ matrix induces a linear transformation from F^n to F^n. First let us assume A is invertible. Therefore, A induces an isomorphism. If the columns of A are not linearly independent then we will find $b_1, \dots, b_n$, not all zero such that $b_1 A_1 + \dots + b_n A_n = 0$, i.e, $A(b_1, \dots, b_n)^t = (0, \dots, 0)^t$. Also, $A(0, \dots, 0)^t = (0, \dots, 0)^t$. This shows that the transformation induced by A is not injective. This is a contradiction and hence the columns of A are linearly independent.

Let us assume the columns of A are linearly independent and hence they form a basis for F^n. Let $e_1, \dots, e_n$ be the standard basis of F^n where e_i is a row vector of length n with 1 at the i^{th} position and zeros elsewhere. Since the columns of A form a basis for F^n, for each e_i there exists $b_{1i}, \dots, b_{ni}$ such that $A_1 b_{1i} + \dots + A_n b_{ni} = e_i$, i.e, $A(b_{1i}, \dots, b_{ni})^t = (e_i)^t$. Consider the matrix

$$B = \begin{pmatrix} b_{11} & \dots & b_{n1} \\ . & \dots & . \\ . & \dots & . \\ b_{n1} & \dots & b_{nn} \end{pmatrix}$$

Note that $AB = I_n$, where I_n is the $n \times n$ identity matrix. Hence A is invertible.